THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2021-22
Lecture 2 practice problems solution
21st January 2022

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **echlam@math.cuhk.edu.hk** if you have any questions.

1. (a) 45° is equal to $\pi/4$ radian. I can turn i into the "origin" of rotation by first taking the translation $T_{-i}(z) = z - i$. And then rotate by $r(z) = e^{i\frac{\pi}{4}}z$, and finally translate back to i: $T_i(z) = z + i$. The total transformation is given by

$$T_i \circ r \circ T_{-i}(z) = e^{i\frac{\pi}{4}}(z - i) + i.$$

   (b) Same as part (a),
$$T(z) = e^{i\frac{\pi}{2}}(z - 2 - i) + 2 + i.$$

   (c) Same as before,
$$T(z) = e^{-i\frac{\pi}{3}}(z - 3) + 3$$

2. We can separate the real and imaginary parts in the equation and obtain $x = \frac{a}{1-c}$ and $y = \frac{b}{1-c}$. Now consider $x^2 + y^2 = \frac{a^2+b^2}{(1-c)^2} = \frac{1-c^2}{(1-c)^2} = \frac{1+c}{1-c} = -1 + \frac{2}{1-c}$. Here we can cancel $1 + c$ because $c = -1$ corresponds to the solve pole, which does not correspond to any number in $\mathbb{C}$. We can then make c the subject,

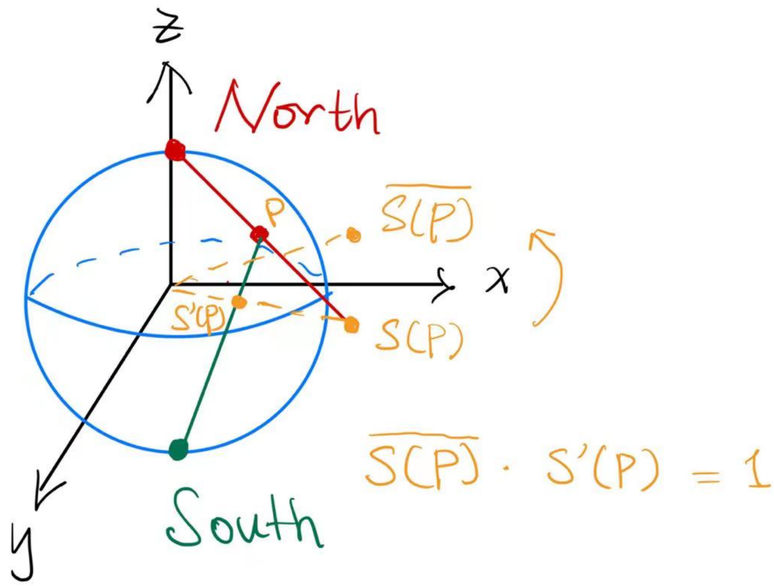
$$c = 1 - \frac{2}{x^2 + y^2 + 1} = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.$$

   And so
$$a = x(1 - c) = \frac{2x}{x^2 + y^2 + 1},$$
$$b = y(1 - c) = \frac{2y}{x^2 + y^2 + 1}.$$

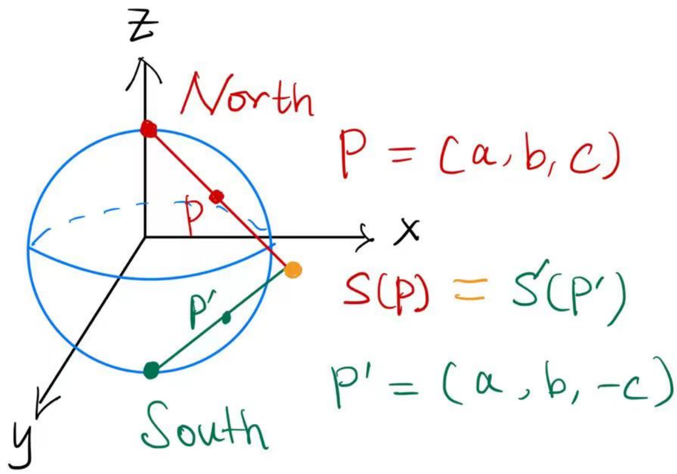
3. (a) Same proof as in lecture 2. We consider the straight line passing through the south pole $(0, 0, -1)$ and also a point (a, b, c) on the sphere. It is given by $t(0, 0, -1) + (1 - t)(a, b, c)$ where t ranges over real numbers. It intersects with the xy-plane at $-t + (1 - t)c = 0$ so $t = \frac{c}{1+c}$. And the intersection point is $(\frac{a}{1+c}, \frac{b}{1+c}, 0) = (x, y, 0)$. So we have
$$x + iy = S'(a, b, c) = \frac{a}{1 + c} + \frac{bi}{1 + c}.$$

(b) By part (a), since (a, b, c) lies on the sphere, we have $a^2 + b^2 = 1 - c^2$, and therefore

$$S(a, b, c) \cdot \overline{S'(a, b, c)} = \frac{a + ib}{1 - c} \cdot \frac{a - ib}{1 + c} = \frac{a^2 + b^2}{1 - c^2} = 1.$$



(c) If we reflect the whole picture along the xy-plane, the sphere undergoes the transformation $T(a, b, c) = (a, b, -c)$. Under this transformation, the north and south poles interchange meanwhile the xy-plane, which we think of as the complex plane is left unchanged. Therefore S and S' interchange. In other words, $S'(a, b, c) = S(T(a, b, c)) = S(a, b, -c)$.



(d) This follows almost directly from part (b), if you can break down the abstract definition of what is meant by lifting. Schematically, we are trying to prove that all the maps fit into the following commutative diagram,

$$\begin{tikzcd}
\mathbb{S}^2 \arrow[r, "{(a,b,c)\mapsto(a,-b,-c)}"] \arrow[d, "S"] & \mathbb{S}^2 \arrow[d, "S"] \\
\hat{\mathbb{C}} \arrow[r, "f(z)=\frac{1}{z}"] & \hat{\mathbb{C}}
\end{tikzcd}$$

Here the top arrow is rotation by $180°$ about the x-axis (it fixes the x-axis, while inverting the signs of y and z coordinates). The vertical arrows are the stereographic projection.

One can prove this by a two steps process, namely we first reflect along the xy-plane on $\mathbb{S}^2$, which has the effect of turning S into S', using the result of part (c). Combined with part (b), we know S and S' are related by $g(z) = 1/\overline{z}$ since $S(a,b,c) \cdot \overline{S'(a,b,c)} = 1$ implies that $S'(a,b,c) = 1/\overline{S(a,b,c)}$. Putting everything together, we obtain

$$S(T(a,b,c)) \overset{\text{part (b)}}{=} S'(a,b,c) \overset{\text{part (c)}}{=} 1/\overline{S(a,b,c)}$$

The second step is to take conjugation $h(z) = \bar{z}$, so that $h(g(z)) = f(z) = \frac{1}{z}$. On the sphere $\mathbb{S}^2$, this amounts to reflecting about xz-plane, because we are taking y to $-y$ in conjugation, without changin x, z coordinates. This gives us the following diagram.

$$\begin{tikzcd}
\mathbb{S}^2 \arrow[r, "{(a,b,c)\mapsto(a,b,-c)}"] \arrow[d, "S"] \arrow[dr, "S'"] & \mathbb{S}^2 \arrow[r, "{(a,b,c)\mapsto(a,-b,c)}"] \arrow[d, "S"] & \mathbb{S}^2 \arrow[d, "S"] \\
\hat{\mathbb{C}} \arrow[r, "g(z)=1/\overline{z}"] & \hat{\mathbb{C}} \arrow[r, "h(z)=\overline{z}"] & \hat{\mathbb{C}}
\end{tikzcd}$$

Geometrically, this just means that we are rotating the sphere $\mathbb{S}^2$, which has the same effect on the complex plane $\hat{\mathbb{C}}$ as taking inversion $\frac{1}{z}$.